ELSEVIER

Chemical Engineering Journal 80 (2000) 73–79

Chemical Engineering Journal

www.elsevier.com/locate/cej

A new method for the reconstruction of the particle radius distribution function from the sedimentation curve

Konstantin V. Parchevsky*

Crimean Astrophysical Observatory, P/O Nauchny, 334413 Crimea, Ukraine

Abstract

A new method for the reconstruction of the particle radius distribution function from the sedimentation curve is proposed. This method permits us to obtain a continuous smooth distribution function. Two approaches are compared. The first approach is based on the calculation of the second derivative from the sedimentation curve. The second one is based on the solution of the original integral equation which describes a sedimentation process. Both of these approaches can be reduced to the problem of the solution of the Fredholm integral equation of the first kind. From the theory of integral equations, it is known that this problem is ill-posed. The usual methods lead to unstable solutions and we are forced to use special regularizing algorithms. In this paper, the Tikhonov regularization method is used to stabilize the solution of the integral equation. It is shown that the accuracy of both methods is higher than the accuracy of the graphical method, but the approach based on the solution of the original integral equation gives a more stable solution than that based on the derivative. The accuracy of the new method permits us to reconstruct the fine structure of the particle radius distribution function. Such an analysis cannot be carried out with the rough bar diagram obtained from the graphical method. The new method is absolutely indispensable in technology for controlling the degree of powder fineness.

Keywords: Sedimentation; Particle radius distribution; Tikhonov regularization; Ill-posed problem

1. Introduction

Let us consider a polydisperse pulp. We introduce the radius distribution function of particles $q(r)$ in such a way that $q(r)\ \mathrm{d}r$ represents the mass fraction of particles with radii in range from r to $r + \mathrm{d}r$ (see Fig. 1). By definition the function $q(r)$ is normalized as follows

$$\int_0^\infty q\,(r)\,\mathrm{d}r = 1 \tag{1}$$

Let us consider the sedimentation of such pulp. Assume that we measure the weight of particles settled on the solid surface at the depth H. One can consider the arbitrary moment of time t. All particles with a settling rate

$$u \geq \frac{H}{t}$$

(or, equivalently, with radii $r \geq r^*$), will settle at this moment of time. Radius r^* is given by the formula

$$r^* = \sqrt{\frac{9\eta H}{2\Delta\rho g t}} \tag{2}$$

We assume that the particles of the pulp settle at a constant rate

$$u = \frac{2}{9}\,\frac{\Delta\rho}{\eta}\,gr^2, \tag{3}$$

where $\Delta\rho$ is the difference between the density of the particle and the liquid, g is the gravitational acceleration, and η is the viscosity of liquid. Then we introduce the mass fraction of totally settled particles Q as follows

$$Q = \int_{r^*}^\infty q\,(r)\,\mathrm{d}r = 1 - \int_0^{r^*} q\,(r)\,\mathrm{d}r \tag{4}$$

Particles with radii $r < r^*$ will settle partially. Let us consider particles with radii from r to $r + \mathrm{d}r$. The mass fraction of these particles is equal to $q(r)\ \mathrm{d}r$ and their sedimentation rate is given by Eq. (3). Only particles which were initially in the column of liquid $h = ut$ in height will sediment at the moment of time t. Thus, the mass fraction of the partially sedimented particles with radii between r and $r + \mathrm{d}r$ equals

$$\mathrm{d}S = \frac{h}{H}\,q\,(r)\,\mathrm{d}r = \frac{2}{9}\,\frac{\Delta\rho}{\eta}\,\frac{g}{H}\,tq\,(r)\,r^2\ \mathrm{d}r$$

The total weight of these particles is given by the finite integral

* Tel.: +38-06554-71619; fax: +38-06554-40704.
E-mail address: kparchevsky@solar.stanford.edu (K.V. Parchevsky).

1383-5866/00/$ – see front matter
PII: S1383-5866(00)00100-3

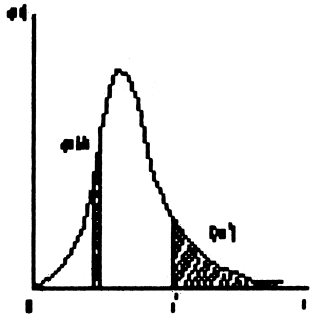

Fig. 1. Normalized radius distribution function of particles: $q(r)\mathrm{d}r$ represents the mass fraction of particles with the radii in range between r and $r+\mathrm{d}r$, $Q(r^*)$ represents the mass fraction of particles with the radii greater than r^*.

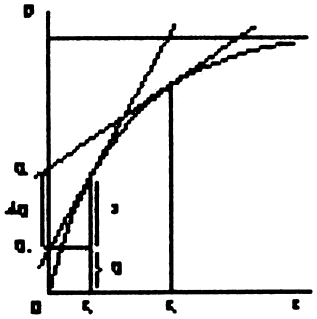

Fig. 2. Sedimentation curve. Tangent line intercepts q_i represent mass fractions of totally sedimented particles at the corresponding moments of time.

$$S = \frac{2}{9}\frac{\Delta\rho}{\eta}\frac{g}{H}t\int_0^{r^*} q\,(r)\,r^2\,\mathrm{d}r \tag{5}$$

The total weight of the sedimented particles is equal to

$$P = Q + S = 1 - \int_0^{r^*} q\,(r)\,\mathrm{d}r + \frac{2}{9}\frac{g\Delta\rho}{\eta H}t\int_0^{r^*} q(r)r^2\mathrm{d}r \tag{6}$$

Introducing the θ-function,

$$\theta\,(z) = \begin{cases} 1 & \text{if} \quad z \geq 0 \\ 0 & \text{if} \quad z < 0 \end{cases}$$

reduces Eq. (6) to a Fredholm integral equation of the first kind

$$\int_0^{\infty} q\,(r)\left(1 - \frac{2\Delta\rho g t r^2}{9\eta H}\right)\theta\left(r^* - r\right)\mathrm{d}r = 1 - P(t) \tag{7}$$

The right-hand side of Eq. (7) is known from the experiment (with some error). One can solve this integral equation directly, but there is another way. Calculating the second derivative from Eq. (7) yields the explicit expression

$$q\left(r^*\right) = -2\frac{t^2}{r^*}\frac{\mathrm{d}^2 P}{\mathrm{d}t^2} \tag{8}$$

The common method of reconstructing the particle size distribution function from the sedimentation curve is the following. One can draw tangent lines to the sedimentation curve (see Fig. 2). The tangent line intercept q represents the mass fraction of totally sedimented particles at the corresponding moment of time t. Hence, the interval $|q_2-q_1|$ represents the mass fraction of liquid-born particles with radii in the range between r_1 and r_2. The particle equivalent radius is given by Eq. (2), where t must be taken equal to the abscissa of tangency point. This procedure is no more than a graphical method of a second order difference calculation. Nothing but a rough bar diagram can be obtained by this method. Also, the graphical method has low accuracy. We need a method which will give a possibility to reconstruct a smooth size distribution function using a sedimentation curve and will be free of the inaccuracy of the graphical method. There are two approaches to the solution of this problem: (a) to solve directly the integral Eq. (7); or (b) to calculate numerically the derivative of the second order from the sedimentation curve and to substitute it in Eq. (8). As we shall see below, both of these approaches are reduced to the solution of a Fredholm integral equation of the first kind. The solution of this equation represents a good example of a so-called ill-posed inverse problem. The ordinary methods do not work here because they lead to unstable solutions and, one way or another, we are forced to use a special regularizing procedure.

2. Statement of the problem

Calculation of the derivative from an experimental function $f(x)$ obtained with errors represents a complicated mathematical problem. The main difficulty is the following. During the experiment, we can measure only a function $f(x)+\varepsilon$, where ε is a noise. The second term does not have a derivative in the classical meaning. If we make an attempt to calculate the derivative by the usual methods, we shall obtain an alternating-sign saw-tooth function of high amplitude and the real derivative $f'(x)$ will 'sink' in the noise generated by the second term. The problem of derivative reconstruction from experimental data is discussed in detail in Parchevsky [1] and Parchevsky and Parchevsky [2]. It is shown that special regularizing algorithms must be used for obtaining a smooth derivative. The appropriate method is the following. One can write an integral equation for the k-th derivative

$$\int_a^b \frac{(x-\xi)^{k-1}}{(k-1)!}\,\theta\,(x-\xi)\,f^{(k)}\,(\xi)\,\mathrm{d}\xi = f(x) - \sum_{j=0}^{k-1}\frac{f^{(j)}\,(a)}{j!}\,(x-a)^j\,, \quad a \leq x \leq b \tag{9}$$

Hence, our problem in both cases is reduced to the solution of a Fredholm integral equation of the first kind. It can be

rewritten in general form as follows

$$Ay \equiv \int_a^b K(x,\,\xi)y(\xi)\mathrm{d}\xi = f(x), \quad c \le x \le d \tag{10}$$

The first equality sign in Eq. (10) can be considered as the definition of the operator A. If we try to solve directly the integral Eq. (7), the kernel $K(x,\xi)$ and the right-hand side $f(x)$ must be chosen as follows:

$$K\,(t,\,r) = \left(1 - \frac{2\Delta\rho g t r^2}{9\eta H}\right)\theta\left(r^* - r\right)$$
$$f\,(t) = 1 - P\,(t)\,. \tag{11}$$

If we want to use Eq. (8), we must calculate the second derivative firstly. To obtain the derivative, we have to solve Eq. (10) with the following kernel and the right-hand side

$$K(x,\,\xi) = (x-\xi)\theta(x-\xi),$$
$$f(x) = P(x) - P(0) - P'(0)x \tag{12}$$

3. The concept of an ill-posed problem

Let us focus our efforts on numerical methods of solution of the integral Eq. (10). Assume that the right-hand side $f(x)$ and the kernel $K(x,\xi)$ are known functions. The kernel $K(x,\xi)$ can be specified either analytically or as a two dimensional array of experimental points. Usually, the right-hand side of Eq. (10) is known from the experiment in the form of N-vector

$$(f\,(x_1)\,,\; f\,(x_2)\,,\; \ldots\,,\; f\,(x_N))^T \tag{13}$$

The most straightforward approach to solve Eq. (10) is the following. One can replace the integral by a finite sum using the quadrature formula of rectangles or trapezoids. Next, instead of the integral Eq. (10) we must solve the system of linear equations

$$\sum_{j=1}^{N} a_{ij}y_j = f_i, \quad i,\; j = 1,\; 2,\; \ldots\,,\; N \tag{14}$$

where a_{ij} are the elements of the square $N \times N$ matrix A which approximates the kernel $K(x,\xi)$, f_i is the vector of the original data, and y_i is the vector of the solution. However, such a simplified attack to the problem does not give an actual solution and only an alternating-sign saw-tooth function can be obtained. Nevertheless, after substituting this solution in the left-hand side of Eq. (10), it coincides with the right-hand side in the first four to sox digits. If we increase N (and the accuracy of the approximation of the integral equation should seem to increase), the solution becomes more and more unstable and the amplitude of fluctuations raises rapidly. The nature of the instability of the solution obtained by applying the quadrature method is the following. It can be shown Verlan and Sizikov [3] that the smallest eigenvalue of the integral operator A (in absolute magnitude) equals zero. When the integral Eq. (10) is approximated by means of the system of linear equations of low order N (e.g. $N < 10$), the eigenvalue spectrum of the integral operator is strongly distorted. The smallest eigenvalue of the matrix of the system (in absolute magnitude) is significantly different from zero. In this case the solution obtained is sufficiently stable but, on the other hand, it is very rough because the order of the system is low. If we increase N, the spectrum of the matrix will approximate the spectrum of the integral operator better and better. The determinant of the matrix will became vanishingly small as the smallest eigenvalue of the matrix goes to zero, and the solution becomes more and more unstable.

It is a vicious circle. If we reduce the degree of an approximation of the integral equation by choosing a low N we can solve the corresponding system of linear equations, but this solution will be far from the actual one due to rough approximation. In this case the spectrum of approximated matrix has nothing in common with the spectrum of the original integral operator. If we try to approximate the original integral equation more precisely we will not be able to solve the corresponding system of linear equations because the determinant of this system will be vanishingly small. We say that the problem of the solution of the integral Eq. (10) is ill-posed.

The concept of ill- (or well-) posed problems goes back to the work of Hadamard [4,5]. The problem is called well-posed if: (1) a solution of the problem exists; (2) the solution is unique; (3) the solution is stable (this means that small variations of the initial data give rise to small variations of the solution). If any one of these conditions is violated the problem will be called ill-posed. The problem of the solution of Eq. (10) is ill-posed because the solution is not stable (item (iii) is violated). Even very small variations of the right-hand side $f(x)$, for example caused by rounding errors, can generate errors in the solution 100-fold greater than the solution itself. Many important applied problems are ill-posed, so new algorithms were developed for their solution.

4. Tikhonov's regularization method

The main idea of this method is the following. We reduce the set of functions among those from which we find the solution of our problem. Let us return to our integral equation and consider the problem of the solution of Eq. (10) by the quadrature method from this point of view. We tried to find any (even discontinuous) solution of the integral equation. We did not impose any constraints on the solution and we have found such a (discontinuous) solution. We did not use all the additional information in the problem statement. If we know that the solution has to be continuous and smooth we must seek it initially among continuous smooth functions.

In general, we can say that the occurrence of the additional a priori information about the solution of the ill-posed problem plays a crucial role during its solution. If we have no additional information about the solution we can obtain nothing but the notorious saw-tooth function as a solution. We can have so much additional information about the solution that the problem can be converted from ill-posed to well-posed. If we know in advance that the solution is represented by an exponential function (for example) we need only fit the coefficients by the least square method. Generally speaking, the more additional information the more stable the methods that can be used to solve the problem.

However, it is dangerous to introduce false a priori information. Trying to make his problem easier to solve the researcher can impose too restrictive constraints without sufficient warranty. If the false additional information is used, one can say in advance that we shall obtain the wrong solution because we solved a completely different problem from that initially posed.

Different methods of regularization require for their implementation different amounts of additional information. Algorithms which require minimum additional information for their implementation are preferable. A new approach, developed by Russian academician Tikhonov (Tikhonov and Arsenin [6]), permits us to obtain regular solutions of the ill-posed problems and requires the minimum a priori information. There is an English translation of this book Tikhonov and Arsenin [7].

Assume, that instead of the exact $K(x, \xi)$ and $f(x)$ we know their approximate values $\tilde{K}(x, \xi)$ and $\tilde{f}(x)$ so that

$$\|\tilde{f}(x) - f(x)\| \leq \delta \tag{15}$$

$$\|\tilde{K}(x, \xi) - K(x, \xi)\| \leq h, \tag{16}$$

i.e. actually we solve the following equation

$$\int_a^b \tilde{K}(x, \xi)\, y(\xi)\, \mathrm{d}\xi = \tilde{f}(x) \tag{17}$$

A tilde is used to designate the values which are known from the experiment. The norm of a function $\|\cdot\|$ can be defined as

$$\|f\| = \left(\int_a^b f^2(\xi)\, \mathrm{d}\xi\right)^{1/2} \tag{18}$$

In functional analysis, functions are considered as 'vectors' of a functional space and the norm of a function is considered as the 'length' of this 'vector'. Therefore, Eq. (15) expresses the fact that the maximal 'distance' between the exact right-hand side and the experimentally measured one does not exceed δ. Similarly, Eq. (16) tells us that the maximal estimation error of the kernel of integral equation does not exceed h. The values of these errors play the role of that additional information which is required for Tikhonov method.

In this algorithm the solution is sought as a function which minimizes the smoothing functional

$$M_\alpha[y] = \|\tilde{A}y - \tilde{f}\|^2 + \alpha\Omega[y] \tag{19}$$

where the positive stabilizing functional $\Omega[y]$ is usually set equal to

$$\Omega[y] = \|y\|^2 \tag{20}$$

and $\alpha > 0$ is the parameter of regularization. It can be proved that the problem of minimizing of Eq. (19) has a unique solution. It is necessary to emphasize that Tikhonov algorithm is no more than a generalization of the least squares method. In fact, we seek the function $y_\alpha(x)$ which minimizes the sum of squared deviations of the right-hand side of Eq. (17) from the left-hand side (i.e. the discrepancy) and has the minimal norm ('length'). This problem on the conditional extremum can be solved by the method of indefinite Lagrangian multipliers and the parameter α is associated with such a multiplier. If we set $\alpha = 0$ we shall return to the usual least squares method (without requirement of minimizing the norm of the solution). If we apply the method discussed above to the solution of integral Eq. (17), we can write Tikhonov and stabilizing functionals as follows

$$M_\alpha[y] = \int_c^d \left(\int_a^b \tilde{K}(x, \xi)\, y(\xi)\mathrm{d}\xi - \tilde{f}(x)\right)^2 \mathrm{d}x + \alpha\Omega[y] \tag{21}$$

$$\Omega[y] = \int_a^b \left(y^2(\xi) + q\left[y'(\xi)\right]^2\right) \mathrm{d}\xi, \quad q \geq 0 \tag{22}$$

The variational problem of the conditional minimization of Eq. (21) can be reduced to the solution of the following Euler equation

$$\alpha\left[y_\alpha(t) - q y''_\alpha(t)\right] + \int_a^b R(t, \xi)\, y_\alpha(\xi)\, \mathrm{d}\xi = \int_c^d \tilde{K}(x, t)\, \tilde{f}(x)\, \mathrm{d}x, \quad a \leq t \leq b \tag{23}$$

$$R(t, \xi) = R(\xi, t) = \int_c^d \tilde{K}(x, t)\, \tilde{K}(x, \xi)\, \mathrm{d}x \tag{24}$$

with the boundary conditions $y'_\alpha(a) = y'_\alpha(b) = 0$. Therefore, instead of the ill-posed Eq. (17), we must solve the well-posed integral (if $q = 0$, zero order of regularization) or integro-differential (if $q \neq 0$, first order of regularization) Eq. (23). Approximating integrals in Eq. (23) by quadrature formulae, we obtain a system of linear equations with a well-conditioned matrix. Parameter q gives us a possibility to control the order of regularization. The increasing of regularization order means imposing stronger restrictions on the smoothness of the solution. The greater order of regularization, the greater smoothness of the solution.

5. Choosing of the regularization parameter

Parameter of regularization α can be chosen in different ways. We shall use the principle of the generalized residual Tikhonov et al. [8]. Introduce the following functions

$$\beta(\alpha) = \|\tilde{A}y_\alpha - \tilde{f}\|^2 \tag{25}$$

$$\gamma(\alpha) = \|y_\alpha\| \tag{26}$$

$$\mu = \inf_y \left(\delta + h\|y\| + \|\tilde{A}y - \tilde{f}\|\right) \tag{27}$$

where y_α is the function which minimizes functional Eq. (21), $\beta(\alpha)$ is the measure of error of our solution. Again, we emphasize that the attempt of direct minimization of Eq. (25) leads to the unstable equations. Value μ gives us a measure of incompatibility of Eq. (17). If an exact solution exists then $\mu = 0$, or else μ is equal to the minimal residual. Function $\gamma(\alpha)$ is the norm ('length') of the solution y_α. Remember that we seek the solution with the minimal norm. Now we are ready to introduce the generalized residual

$$\rho(\alpha) = \beta(\alpha) - (\delta + h\gamma(\alpha) + \mu)^2 \tag{28}$$

where δ and h are errors of the right-hand side and the kernel correspondingly. In accordance with the principal of generalized residual, the optimal value of the regularization parameter α^* can be found from the nonlinear equation

$$\rho(\alpha^*) = 0 \tag{29}$$

The meaning of Eq. (29) is the following. We choose α^* in such a way that the residual $\beta(\alpha)$ becomes comparable with the total error of the method, which consists of the right-hand side error δ, the error of the kernel approximation h and the incompatibility of the initial equation μ.

6. Numerical realization of the algorithm

Let us consider the numerical details of the algorithm realization. The solution of Eq. (17) can be obtained in two passes. At the first pass, the measure of incompatibility μ is calculated. At the second pass, the nonlinear Eq. (29) is solved to find optimal parameter of regularization. At both passes, it is required to minimize the functional $M_\alpha[y]$ (Eq. (21)) for different values of α by solving the integro-differential Eq. (23). Approximating the integrals in Eq. (23) by the trapezoid formula with variable step and representing the second derivative by finite difference, one can obtain the following system of linear equations

$$\left(G_{ij} + \alpha C_{ij}\right) y_i^{(\alpha)} = F_j \tag{30}$$

with the symmetrical positive definite matrix G_{ij} and tridiagonal banded matrix C_{ij}. There are many efficient algorithms to solve such system. In our program we used Cholesky square root method Collection [9]. This standard subroutine was translated from the FORTRAN language to C++.

At the first pass, the functional

$$\Phi_{\delta,h}[y] = \delta + h\|y\| + \|\tilde{A}y - \tilde{f}\| \tag{31}$$

must be minimized to calculate the measure of incompatibility $\mu = \min \Phi[y]$. It is shown [8] that the problem of minimizing Eq. (31) is equivalent to the problem of minimizing $M_\alpha[y]$ if we choose α so that the function

$$\psi(\alpha) = \|\tilde{A}y_\alpha - \tilde{f}\| + h\|y_\alpha\| \tag{32}$$

will take on a minimum value. In this program the golden-section method was used to minimize Eq. (32). At first, the interval which contains a minimum is roughly estimated, then the exact position of this minimum is found by the iterative refinement of the interval in the golden ratio. The function $\psi(\alpha)$ is calculated at each step as follows. For a given α, the function y_α which minimizes the functional $M_\alpha[y]$ is found by solving the system of linear Eq. (30). Next, this function is substituted to the right-hand side of Eq. (32) to find a value of $\psi(\alpha)$.

At the second pass, nonlinear Eq. (29) is solved by the chord method using μ calculated at the first step. The initial values of α_0 and α_1 can be taken in such a way that $\rho(\alpha_0)$ and $\rho(\alpha_1)$ are positive and $\rho(\alpha_0) > \rho(\alpha_1)$. Such initial values can always be found. In this case, the sequence of α_n obtained during iterations of the chord method

$$\lambda_0 = \lambda_0 - \frac{(\lambda_0 - \lambda_1)\,\rho(1/\lambda_0)}{\rho(1/\lambda_0) - \rho(1/\lambda_1)}, \quad \alpha_n = \frac{1}{\lambda_n},$$
$$\lambda_0 = \lambda_1, \quad \lambda_1 = \lambda_n, \tag{33}$$

is monotonic, the residual $\rho(1/\lambda_n)$ is positive for all n and the sequence converges. If $h \geq 0$ the convexity of $\rho(\alpha)$ can be violated. It means that for some terms α_n the residual can be negative, i.e. $\rho(\alpha_n) < 0$. In this case we must use not the chord method, but the modified chord method:

$$\lambda_n = \lambda_0 - \frac{(\lambda_0 - \lambda_1)\,\rho(1/\lambda_0)}{\rho(1/\lambda_0) - \rho(1/\lambda_1)}, \quad \alpha_n = \frac{1}{\lambda_n},$$
$$\text{if } \rho(1/\lambda_0)\,\rho(1/\lambda_n) < 0, \quad \text{then } \lambda_1 = \lambda_n, \tag{34}$$
$$\text{if } \rho(1/\lambda_1)\,\rho(1/\lambda_n) < 0, \quad \text{then } \lambda_0 = \lambda_n.$$

Satisfaction of one of these inequalities can be guaranteed. Each iteration of the chord method begins with the calculation of $\rho(\alpha_n)$. For a given α_n, the minimum of the functional $M_\alpha[y]$ is found by solving Eq. (30), next, this solution is substituted in the right-hand side of Eq. (28) to find the residual. The algorithm, described above, was realized as a program in Watcom C++ 10.0.

7. Results and discussion

The numerical simulation was carried out to compare the accuracy of two different methods of the reconstruction of the radius distribution function. An artificial

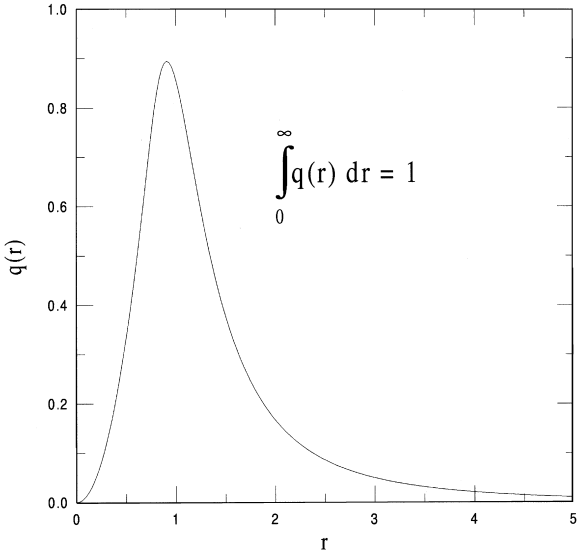

Fig. 3. Artificial normalized radius distribution function of the particles.

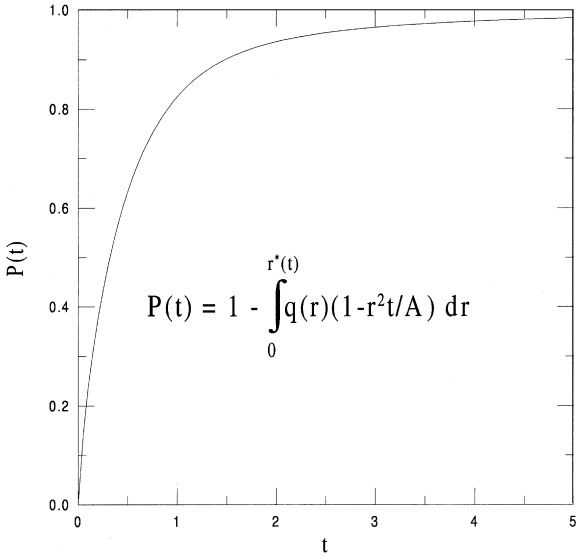

Fig. 4. Simulated sedimentation curve. The system of units is chosen in such a way that $A = 1$.

non-symmetrical radius distribution function was chosen as follows (Fig. 3)

$$q\,(r) = \frac{r^2}{0.739386}\left(1 - \exp\left(-\frac{1}{r^5}\right)\right). \tag{35}$$

The normalized coefficient was chosen in such a way that Eq. (1) is satisfied. The sedimentation curve of the particles for a given radius distribution function can be easily obtained from Eq. (6)

$$P\,(t) = 1 - \int_0^{r^*(t)} q\,(r)\,(1 - r^2 t)\mathrm{d}r \tag{36}$$

The term $2/9\ g\Delta\rho/\eta H$ was set equal to unity to make the values t and r of order unity. This is needed to avoid a sacrifice of accuracy during the numerical calculations. Such transformation always can be done by the appropriate choice of units. The sedimentation curve, calculated by means of Eq. (36), is shown in Fig. 4. Adding noise to the calculated sedimentation curve simulates experimental errors. The direct problem of obtaining the sedimentation curve from the radius distribution function is much more easier than the inverse one. Now we are ready to solve the ill-posed inverse problem of the reconstruction of the radius distribution function from the sedimentation curve.

We shall start from approach (b) based on the calculation of the second derivative from the sedimentation curve Eq. (8) (see Section 1). The results of this calculation are shown in Fig. 5. The second derivative was calculated by means of the solution of the integral Eq. (10) where the kernel and the right-hand side are given by Eq. (12). Tikhonov regularization method was used to stabilize the solution. In general, this method reconstructs the original distribution with satisfactory accuracy, but, in the range of small equivalent radii, nonphysical oscillations appear. They are caused by noise added to the sedimentation curve. Information about the first derivative (from the sedimentation curve) at the origin is needed to calculate the right-hand side of the integral equation for the second derivative (see Eq. (12)). It can be found, for example, by means of the first difference. Of course, we could apply Tikhonov method to find the first derivative and then to use its value at zero in the future, but at the bounds of the interval the derivative is reconstructed with big errors and it was found that using the first difference is preferable. Uncertainty in $P'(0)$ strongly affects the appearance of the nonphysical oscillations in the solution. Method (a) based on the direct solution of the integral Eq. (7) is free from such

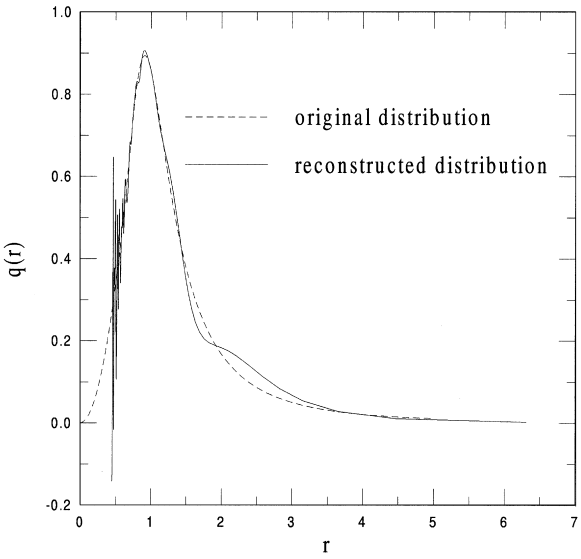

Fig. 5. The equivalent radius distribution function reconstructed by calculating the second derivative. Nonphysical oscillations at the beginning of the curve are caused by noise added to the sedimentation curve to simulate instrumental errors.

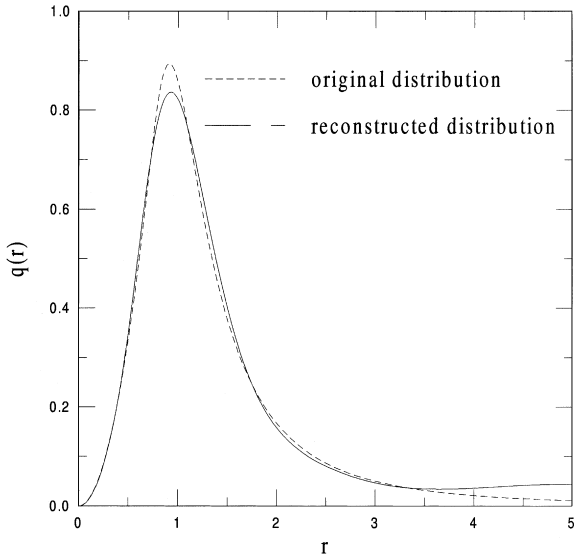

Fig. 6. Radius distribution function reconstructed by means of direct solution of the original integral equation. This approach gives smooth curve free of nonphysical oscillations.

uncertainty. It was shown that in both cases we must solve the integral equation. Hence, using the 'explicit' equation formula 8 does not give us any advantage and direct solution of the original integral Eq. (7) is more preferable.

The radius distribution curve reconstructed by solving the original integral Eq. (7) is shown in Fig. 6. It is clear that this approach is much better than that based on the derivative reconstruction. The solution is completely free from the nonphysical oscillations, and is smooth and stable. Some divergence in the range of big equivalent radii can be described as follows. This part of the curve is formed by quickly sedimenting particles with big radii. The existence of a minimal time sampling interval results in finite resolution in the equivalent radii. The time sampling interval at the beginning of the sedimentation curve is equal to $\Delta t = 0.05$. This means that, during the period from the beginning of the experiment to the first measurement, all particles with radii $r > r^*$ will sediment. Remember that in the chosen system of units, sediment time and equivalent radius are related by the following formula

$$r^* = \sqrt{1/t}.$$

Hence, from such 'measurements' (remember that we use the artificial data), we cannot obtain information about particles with radii $r > r^* \approx 4.5$. Actually, the restriction is more stronger. The first measurement gives us the sedimented weight of all particles with radii greater than r^*. Only the second measurement gives us the weight of particles with radii in the range between r and $r + \Delta r$. So we need at least two measurements to obtain significant information for the reconstruction of the radius distribution function of particles with big radii. Hence, the maximum reliable equivalent radius is the following

$$r_{\max} \approx \sqrt{\frac{1}{2\Delta t}}.$$

In our case, $r_{\max} \approx 3.2$.

8. Conclusion

We carried out a comparison of two methods which reconstruct the radius distribution function from the sedimentation curve. The first method is based on the calculation of the second derivative from the sedimentation curve. The second one is based on the solution of the original integral equation which describes the sedimentation process. It is shown that this problem is ill-posed. Both of these algorithms use the regularizing procedure for stabilizing the solution. In both cases the regularization is based on Tikhonov method. It is shown that the approach based on the solution of the original integral equation is much more stable than that based on the derivative reconstruction.

This method can be used for reconstructing a radius distribution function with the several peaks as well. The package of computer programs which implement this method can be used as a software component of the monitoring system of the quality of powder fineness in various flow processes.

References

[1] K.V. Parchevsky, Ecol. Model. 133 (1–2) (2000) 107–115.
[2] K.V. Parchevsky, V.P. Parchevsky, Thermochim. Acta 309 (1998) 181.
[3] A.F. Verlan, V.S. Sizikov, Integral equations: methods, algorithms, programs, Naukova Dumka, Kiev, 1986 (in Russian).
[4] J. Hadamard, Bull. Univ. Princeton 13 (1902) 49.
[5] J. Hadamard, Le problème de Caucy et les équations aux derivees partielles linéaires huperboliques, Hermann, Paris, 1932.
[6] A.N. Tikhonov, V.J. Arsenin, Methods of the solution of ill-posed problems, Nauka, Moscow, 1979 (in Russian).
[7] A.N. Tikhonov, V.J. Arsenin, Solutions of ill-posed problems, Winston and Sons, Washington, 1977.
[8] A.N. Tikhonov, A.V. Goncharsky, V.V. Stepanov, A.G. Jagola, Numerical methods for the solution of ill-posed problems, Nauka, Moscow, 1990 (in Russian).
[9] Collection of the scientific subroutines in FORTRAN. Issue 2. Matrix algebra and linear algebra, Statistika, Moscow, 1974 (in Russian).